Assignment 4.

This homework is due *Thursday*, October 1.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. Exercises

(1) Let $a_n \in \mathbb{R}$, $n \in \mathbb{N}$. Prove that if $\sum_{n=1}^{\infty} |a_n|$ is summable then so is $\sum_{n=1}^{\infty} a_n$. Show that the converse is false.

(2) (1.4.47) Let E be a closed set of real numbers and $f : E \to \mathbb{R}$ be continuous. Show that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g|_E = f$. (*Hint:* Take g to be linear on each of the intervals of which $\mathbb{R} \setminus E$ is composed.)

(3) ($\sim$1.4.49) Let $f, g : E \to \mathbb{R}$ be continuous.
   (a) Let $\max\{f, g\} : E \to \mathbb{R}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, $x \in E$. Show that that $\max\{f, g\}$ is continuous.
   (b) Show that $|f|$ is continuous.

   (*Hint:* Show that $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ and $|a| = \max\{a, -a\}$. Conclude that it is enough to prove one of the above statements.)

(4) (1.4.51) (Approximation of continuous functions by piecewise linear ones) A continuous function φ on $[a, b]$ is called *piecewise linear* provided there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ for which φ is linear on each interval $[x_i, x_{i+1}]$.

   Let f be continuous on $[a, b]$ and ε a positive number. Show that there is a piecewise linear function φ on $[a, b]$ with $|f(x)-\varphi(x)| < \varepsilon$ for all $x \in [a, b]$. (*Hint:* Use uniform continuity.)

(5) (Brouwer theorem for a segment) Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f([0, 1]) \subseteq [0, 1]$ (i.e., all values of f are contained in $[0, 1]$). Then there is a point $x \in [0, 1]$ such that $f(x) = x$.

(6) (a) On an infinite frying pan, there is a bounded pancake[1]. Prove that one can make a straight cut in any given direction (that is, parallel to a given line) that splits the pancake in halves of equal area. (*Hint:* Use intermediate value theorem.)
   (b) On an infinite frying pan, there are two bounded pancakes. Prove that one can make a straight cut that splits *each* pancake into halves of equal area.

(7) (1.4.52) Show that a nonempty subset E of $\mathbb{R}$ is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

(8) (1.4.58) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the inverse image w.r.t. f of every closed set is closed, and of every Borel set is Borel. (*Hint:* Show that f^{-1} respects set-theoretic operations. For the last part of the problem, consider the set $\mathcal{A}$ of subsets E such that $f^{-1}(E)$ is Borel.)

[1] A subset that has area. If you are too worried about details, you may think it's a polygon.